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Orthomodular Implication Algebras[1]

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In Boolean algebras the properties of the implication operation can be modeled by a so-called implication algebra that itself can be considered as a join-semilattice with 1 whose principal filters are Boolean algebras. This situation is generalized from Boolean algebras to orthomodular lattices.

The classical two-valued propositional logic has its algebraic counterpart in a Boolean algebra. If one considers only the logical connective implication of classical logic then the clone generated by this connective is not the clone of all Boolean functions. The algebraic counterpart of the mentioned case is the so-called implication algebra introduced and treated by Abbott (1967). Similarly, an algebraic counterpart of the fragment of intuitionistic logic containing only intuitionistic implication and the constant 1 (which can serve as a true value) was introduced by Henkin and treated by Diego (1967) under the name Hilbert algebra.

In some considerations concerning quantum mechanics, another type of logic turned out to be suitable. It was investigated by numerous authors under various names (cf. e.g., Beran, 1984; Burmeister and Mączyński, 1994; Dorninger *et al.*, 2000; Kalmbach, 1983; Pulmannová, 1993). However, algebraic counterparts of these logics are either orthomodular lattices (cf. e.g., Beran, 1984; Kalmbach, 1983) or the so-called orthomodular algebras (Burmeister and Mączyński, 1994) or certain generalizations of Boolean rings (cf. e.g., Dorninger *et al.*, 2000). These logics are in some sense correlated with the Hilbert space logic of quantum

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mechanics but there are still some differences concerning the interpretation. This fact motivated us to consider only the reduct of this so-called orthomodular logic that contains the connective implication and we found its algebraic representation. We believe that this attempt can be helpful for providing a better interpretation of the connective of implication in quantum mechanical logic.

In classical logic the implication $\rightarrow$ can be composed by means of disjunction $\vee$ and negation $'$ as follows: $x \rightarrow y := x' \vee y$. If instead of a Boolean algebra an orthomodular lattice $\mathcal{L}$ is considered then there exist exactly six binary terms coinciding with the usual implication operation in the case that $\mathcal{L}$ is a Boolean algebra. It turns out that for our purposes the most useful term representing implication in orthomodular lattices is $(x' \wedge y') \vee y$.

Lemma 1. *In a Boolean algebra* $(B, \cup, \cap, ', 0, 1)$ *the implication operation* $\rightarrow$ *defined by*

$$x \rightarrow y := x' \cup y$$

for all $x, y \in B$ *satisfies the following axioms:*

$$x \rightarrow x = 1,$$
$$(x \rightarrow y) \rightarrow x = x,$$
$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x,$$

and

$$x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z).$$

Proof: Easy. □

These axioms seem to be the characterizing properties of the connective of implication in Boolean algebras, i.e. in classical logic. This was the motivation for the following definition:

Definition 1. (Abbott, 1967). An *implication algebra* is an algebra $(A, \cdot, 1)$ of type (2, 0) satisfying the following axioms:

(I1) $xx = 1$,
(I2) $(xy)x = x$,
(I3) $(xy)y = (yx)x$, and
(I4) $x(yz) = y(xz)$.

Remark 1. Originally (Abbott, 1967) an implication algebra was defined as a groupoid satisfying the axioms (I2)–(I4). That the term xx is constant follows from

(I2) and (I3) (cf. Abbott, 1967):

$$\begin{aligned}
xx &= ((xy)x)x = (x(xy))(xy) = (((xy)x)(xy))(xy) = (xy)(xy)\\
&= (((xy)y)(xy))(xy) = ((xy)((xy)y))((xy)y)\\
&= ((((xy)y)(xy))((xy)y))((xy)y) = ((xy)y)((xy)y)\\
&= ((yx)x)((yx)x) = ((((yx)x)(yx))((yx)x))((yx)x) = ((yx)((yx)x))((yx)x)\\
&= (((yx)x)(yx))(yx) = (yx)(yx) = (((yx)y)(yx))(yx) = (y(yx))(yx)\\
&= ((yx)y)y = yy.
\end{aligned}$$

One can ask about the structure of implication algebras. A close inspection shows that these algebras can be considered as very special posets that are set-theoretical unions of Boolean algebras. Theorem 1 describes the structure of implication algebras as so-called Boolean join-semilattices.

Definition 2. A *Boolean join-semilattice* is an algebra of the form $(A, \cup, 1, ({}^p; p \in A))$ where $(A, \cup, 1)$ is a join-semilattice with greatest element 1 and for each $p \in A$, ^{p} is a unary operation on $[p, 1]$ such that for each $p \in A$, $([p, 1], \cup, \cap_p, {}^p, p, 1)$ is a Boolean algebra where $\cap_p$ denotes the meet-operation corresponding to the partial order induced by $\cup$.

It was proved in Abbott (1967).

Theorem 1. *If* $\mathcal{A} = (A, \cdot, 1)$ *is an implication algebra and one defines*

$$x \vee y := (xy)y \text{ for all } x,\ y \in A$$

and

$$x^p := xp \text{ for all } p \in A \text{ and all } x \in [p,\ 1]$$

then $\mathbf{S}(\mathcal{A}) := (A, \vee, 1, ({}^p; p \in A))$ *is a Boolean join-semilattice. Conversely, if* $\mathcal{S} = (A, \cup, 1, ({}^p; p \in A))$ *is a Boolean join-semilattice and one defines*

$$xy := (x \cup y)^y$$

for all $x,\ y \in A$ *then* $\mathbf{A}(\mathcal{S}) := (A, \cdot, 1)$ *is an implication algebra. For every fixed base set A the mappings* **S** *and* **A** *are mutually inverse bijections between the set of all implication algebras over A and the set of all Boolean join-semilattices over A.*

Moreover, in every implication algebra the partial order relation is compatible with the binary operation in the following way:

Lemma 2. (Abbott, 1967). *If* $(A, \cdot, 1)$ *is an implication algebra then* $x, y, z \in A$ *and* $x \leq y$ *together imply both* $zx \leq zy$ *and* $xz \geq yz$.

Definition 3. An *orthomodular lattice* is an algebra $(L, \cup, \cap, {}', 0, 1)$ of type (2, 2, 1, 0, 0) such that $(L, \cup, \cap, 0, 1)$ is a bounded lattice and the following conditions are satisfied:

(i) $x \cup x' = 1$,
(ii) $(x \cup y)' = x' \cap y'$,
(iii) $(x')' = x$, and
(iv) $x \leq y$ implies $y = x \cup (y \cap x')$.

It is well known that an orthomodular lattice is a Boolean algebra if and only if it is distributive and that in the variety of orthomodular lattices there exist exactly six terms that coincide with the usual implication term in the case of Boolean algebras. It turns out that the following term is appropriate for our purposes:

$$(x' \cap y') \cup y.$$

The following lemma shows that similar to the results in Lemma 1, the implication operation just defined has some natural properties:

Lemma 3. *In an orthomodular lattice* $(L, \cup, \cap, ', 0, 1)$ *the implication operation* $\rightarrow$ *defined by*

$$x \rightarrow y := (x' \cap y') \cup y$$

for all $x, y \in L$ *satisfies the following axioms:*

$$x \rightarrow x = 1,$$
$$x \rightarrow (y \rightarrow x) = 1,$$
$$(x \rightarrow y) \rightarrow x = x,$$
$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x,$$
$$(((x \rightarrow y) \rightarrow y) \rightarrow z) \rightarrow (x \rightarrow z) = 1,$$

and

$$((((((((x \rightarrow y) \rightarrow y) \rightarrow z) \rightarrow z) \rightarrow z) \rightarrow x) \rightarrow x) \rightarrow z) \rightarrow x) \rightarrow x$$
$$= (((x \rightarrow y) \rightarrow y) \rightarrow z) \rightarrow z.$$

Moreover,

$$(x \rightarrow y) \rightarrow y = x \cup y$$

for all $x, y \in L$.

Proof: Let $(L, \cup, \cap, ', 0, 1)$ be an orthomodular lattice and define $x \rightarrow y := (x' \cap y') \cup y$ for all $x, y \in L$. Then for all $x, y, z \in L$ it holds:

$$(x \rightarrow y) \rightarrow y = (((x' \cap y') \cup y)' \cap y') \cup y = ((x \cup y) \cap y') \cup y = x \cup y,$$
$$x \rightarrow x = (x' \cap x') \cup x = 1,$$

$$
\begin{aligned}
x \to (y \to x) &= (x' \cap ((y' \cap x') \cup x)') \cup (y' \cap x') \cup x \\
&= (x' \cap (y \cup x)) \cup (x' \cap y') \cup x \\
&= ((x \cup y) \cap x') \cup (x' \cap y') \cup x = 1, \\
(x \to y) \to x &= (((x' \cap y') \cup y)' \cap x') \cup x = ((x \cup y) \cap y' \cap x') \cup x \\
&= 0 \cup x = x, \\
(x \to y) \to y &= x \cup y = y \cup x = (y \to x) \to x,
\end{aligned}
$$

$$
\begin{aligned}
&(((x \to y) \to y) \to z) \to (x \to z) \\
&\quad = ((((x \cup y)' \cap z') \cup z)' \cap ((x' \cap z') \cup z)') \cup (x' \cap z') \cup z \\
&\quad = ((x \cup y \cup z) \cap z' \cap (x \cup z) \cap z') \cup (x' \cap z') \cup z \\
&\quad = ((x \cup z) \cap z') \cup (x' \cap z') \cup z = 1
\end{aligned}
$$

and

$$
\begin{aligned}
&(((((((((x \to y) \to y) \to z) \to z) \to z) \to x) \to x) \to z) \to x) \to x \\
&\quad = ((((x \cup y \cup z)' \cap z') \cup z \cup x)' \cap z') \cup z \cup x \\
&\quad = ((x \cup y \cup z) \cap z' \cap x') \cup z \cup x \\
&\quad = ((x \cup y \cup z) \cap x' \cap z') \cup x \cup z = x \cup y \cup z \\
&\quad = (((x \to y) \to y) \to z) \to z. \quad \square
\end{aligned}
$$

Again one may ask whether the properties just considered are necessary for describing the implication reduct of orthomodular logics. The affirmative answer leads to the following definition:

Definition 4. An *orthomodular implication algebra* is an algebra $(A, \cdot, 1)$ of type (2, 0) satisfying the following axioms:

(O1) $xx = 1$,
(O2) $x(yx) = 1$,
(O3) $(xy)x = x$,
(O4) $(xy)y = (yx)x$,
(O5) $(((xy)y)z)(xz) = 1$, *and*
(O6) $(((((((((xy)y)z)z)z)x)x)z)x)x = (((xy)y)z)z$.

In an orthomodular implication algebra $(A, \cdot, 1)$ a binary relation $\leq$ and a binary operation $\vee$ are defined by

$$x \leq y \text{ if and only if } xy = 1$$

and

$$x \vee y := (xy)y,$$

respectively, for all $x, y \in A$.

Remark 2. Using the just defined binary operation $\vee$ (O4)–(O6) can be written more shortly as follows:

(O4) $x \vee y = y \vee x$,
(O5) $((x \vee y)z)(xz) = 1$, *and*
(O6) $(((x \vee y) \vee z)z \vee x)z \vee x = (x \vee y) \vee z$.

The following lemma shows that orthomodular implication algebras are also very special posets:

Lemma 4. *For an orthomodular implication algebra* $(A, \cdot, 1)$ *and* $a, b, c \in A$ *the following hold:*

(i) $a \leq 1$,
(ii) $1a = a$,
(iii) $a \leq b$ *if and only if* $a \vee b = b$,
(iv) $a \vee 1 = 1$,
(v) $a \leq a$,
(vi) $a \leq b \leq a$ *implies* $a = b$,
(vii) $a \leq b \leq c$ *implies* $a \leq c$,
(viii) $a \leq a \vee b$,
(ix) $a \leq b$ *implies* $bc \leq ac$, *and*
(x) $a, b \leq c$ *implies* $a \vee b \leq c$.

Proof:

(i) $a1 = a(aa) = 1$ according to (O1) and (O2) and hence $a \leq 1$.
(ii) $1a = (aa)a = a$ according to (O1) and (O3).
(iii) If $a \leq b$ then $a \vee b = (ab)b = 1b = b$ according to (ii). If, conversely, $a \vee b = b$ then $ab = a(a \vee b) = a(b \vee a) = a((ba)a) = 1$ according to (O4) and (O2) and hence $a \leq b$.
(iv) $a \vee 1 = 1 \vee a = (1a)a = aa = 1$ according to (O4), (ii), and (O1).
(v) $aa = 1$ according to (O1) and hence $a \leq a$.
(vi) $a \leq b \leq a$ implies $a = 1a = (ba)a = b \vee a = a \vee b = (ab)b = 1b = b$ according to (ii) and (O4).
(vii) $a \leq b \leq c$ implies $1 = bc = (a \vee b)c \leq ac \leq 1$ according to (iii), (O5), and (i) whence by (vi) $ac = 1$, i.e. $a \leq c$.
(viii) $a(a \vee b) = a(b \vee a) = a((ba)a) = 1$ according to (O4) and (O2) and hence $a \leq a \vee b$.

(ix) $a \le b$ implies $(bc)(ac) = ((a \vee b)c)(ac) = 1$ according to (iii) and (O5) and hence $bc \le ac$.

(x) If $a, b \le c$ then $cb \le ab$ according to (ix) whence $a \vee b = (ab)b \le (cb)b = c \vee b$ by (ix). This shows $(a \vee b)c = (a \vee b)(b \vee c) = (a \vee b)(c \vee b) = 1$ according to (iii) and (O4) whence $a \vee b \le c$. □

A close inspection of the local behavior of orthomodular implication algebras leads to the following definition:

Definition 5. An *orthomodular join-semilattice* is an algebra of the form $(A, \vee, 1, (^p; p \in A))$ where $(A, \vee, 1)$ is a join-semilattice with greatest element 1 and for each $p \in A$, ^{p} is a unary operation on $[p, 1]$ such that for each $p \in A$, $([p, 1], \vee, \wedge_p, ^p, p, 1)$ is an orthomodular lattice where $\wedge_p$ denotes the meet-operation corresponding to the partial order induced by $\vee$.

Theorem 2. *If* $\mathcal{A} = (A, \cdot, 1)$ *is an orthomodular implication algebra and one defines*

$$x \vee y := (xy)y \text{ for all } x, y \in A$$

and

$$x^p := xp \text{ for all } p \in A \text{ and all } x \in [p, 1]$$

then $\mathbf{S}(\mathcal{A}) := (A, \vee, 1, (^p; p \in A))$ *is an orthomodular join-semilattice.*

Proof: Let $(A, \cdot, 1)$ be an orthomodular implication algebra. From (v), (vi), (vii), and (i) of Lemma 4 it follows that $(A, \le)$ is a poset with greatest element 1. (O4) together with (viii) and (x) of Lemma 4 imply that $\vee$ is the join-operation corresponding to $\le$. Hence $(A, \vee, 1)$ is a join-semilattice with greatest element 1. Now fix $p \in A$ and put $x^p := xp$ for all $x \in A$. From (ix) and (viii) of Lemma 4 it follows that $(^p, ^p)$ is a Galois connection between $(A, \le)$ and $(A, \le)$. Since $xp \ge p$ for all $x \in A$ according to (O2) and since $x \in A$ and $x \ge p$ together imply $x = x \vee p = (xp)p$, it follows $A^p = Ap = [p, 1]$. Hence $^p|[p, 1]$ is an involutory antiautomorphism of $([p, 1], \le)$. This clearly implies $(x \vee y)^p = x^p \wedge_p y^p$ for all $x, y \in [p, 1]$ if one defines $x \wedge_p y := (x^p \vee y^p)^p$ for all $x, y \in [p, 1]$ and it also follows that $\wedge_p$ is the meet-operation corresponding to $\le$. Moreover,

$$x \vee x^p = x^p \vee x = ((xp)x)x = xx = 1$$

for all $x \in A$ according to (O3) and (O1). Finally, for $x, y \in [p, 1]$ with $x \le y$ it holds

$$\begin{aligned} y &= x \vee y \vee p = ((x \vee y \vee p)p \vee x)p \vee x = (yp \vee x)p \vee x \\ &= (y^p \vee x)^p \vee x = (y \wedge_p x^p) \vee x = x \vee (y \wedge_p x^p) \end{aligned}$$

according to (O6), which proves that $([p, 1], \vee, \wedge_p, {}^p, p, 1)$ is an orthomodular lattice. □

Theorem 3. *If* $\mathcal{S} = (A, \cup, 1, ({}^p; p \in A))$ *is an orthomodular join-semilattice and one defines*

$$xy := (x \cup y)^y$$

for all $x, y \in A$ *then* $\mathbf{A}(\mathcal{S}) := (A, \cdot, 1)$ *is an orthomodular implication algebra.*

Proof: Let $(A, \cup, 1, ({}^p; p \in A))$ be an orthomodular join-semilattice. Then for all $x, y, z \in A$ it holds:

$$x \vee y := (xy)y = ((x \cup y)^y \cup y)^y = ((x \cup y)^y)^y = x \cup y,$$

(O1) $xx = (x \cup x)^x = x^x = 1,$
(O2) $x(yx) = (x \cup (y \cup x)^x)^{(y \cup x)^x} = ((y \cup x)^x)^{(y \cup x)^x} = 1,$
(O3) $(xy)x = ((x \cup y)^y \cup x)^x = ((x \cup y)^y \cup y \cup x)^x = 1^x = x,$
(O4) $x \vee y = x \cup y = y \cup x = y \vee x,$
(O5) $((x \vee y)z)(xz) = ((x \cup y \cup z)^z \cup (x \cup z)^z)^{(x \cup z)^z} = ((x \cup z)^z)^{(x \cup z)^z} = 1$, and
(O6) $(((x \vee y) \vee z)z \vee x)z \vee x = ((x \cup y \cup z)^z \cup x \cup z)^z \cup x$

$$= ((x \cup y \cup z) \cap (x \cup z)^z) \cup x \cup z = x \cup y \cup z$$
$$= (x \vee y) \vee z. \quad \square$$

Theorem 4. *For fixed base set A the mappings* **S** *and* **A** *are mutually inverse bijections between the set of all orthomodular implication algebras over A and the set of all orthomodular join-semilattices over A.*

Proof: If $\mathcal{A} = (A, \cdot, 1)$ is an orthomodular implication algebra, $\mathbf{S}(\mathcal{A}) = (A, \vee, 1, ({}^p; p \in A))$ and $\mathbf{A}(\mathbf{S}(\mathcal{A})) = (A, \circ, 1)$ then

$$x \circ y = (x \vee y)^y = ((xy)y)y = xy \vee y = xy$$

for all $x, y \in A$ according to (O2). If, conversely, $\mathcal{S} = (A, \vee, 1, ({}^p; p \in A))$ is an orthomodular join-semilattice, $\mathbf{A}(\mathcal{S}) = (A, \cdot, 1)$ and $\mathbf{S}(\mathbf{A}(\mathcal{S})) = (A, \cup, 1, ({}_p; p \in A))$ then

$$x \cup y = (xy)y = ((x \vee y)^y \vee y)^y = ((x \vee y)^y)^y = x \vee y$$

for all $x, y \in A$ and

$$x_p = xp = (x \vee p)^p = x^p$$

for all $p \in A$ and all $x \in [p, 1]$. □

Corollary 1 (Abbott, 1967). *The orthomodular join-semilattice corresponding to an implication algebra is a Boolean join-semilattice. The orthomodular implication algebra corresponding to a Boolean join-semilattice is an implication algebra.*

Remark 3. Theorem 4 and Corollary 1 say that the mappings **S** and **A** are extended from the “Boolean case” to the “orthomodular case.”

Corollary 2. *Every implication algebra is an orthomodular implication algebra. An orthomodular implication algebra is an implication algebra if and only if it satisfies the condition* (I4).

In contrast to Lemma 2 saying that in implication algebras the partial order relation is compatible with the binary operation in a certain sense, in orthomodular implication algebras a similar result can be obtained only for special triples of elements:

Lemma 5. *Let* $(A, \cdot, 1)$ *be an orthomodular implication algebra and* $a, b, c \in A$. *Then* (i) *and* (ii) *hold:*

(i) *If* $a \leq c$ *or* $b \leq c$ *then* $a(bc) = b(ac)$.
(ii) *If* $a, c \leq b$ *then* $ca \leq cb$.

Proof:

(i) If $a \leq c$ then $ac = 1$ and hence $b(ac) = b1 = 1$. Moreover, $a \leq c \leq bc$. Therefore also $a(bc) = 1$. The case $b \leq c$ is symmetric to the case $a \leq c$.
(ii) If $a, c \leq b$ then $ca \leq 1 = cb$. □

Lemma 6. *If* $(L, \cup, \cap, ', 0, 1)$ *is an orthomodular lattice and* $a, b, c \in L$ *with* $a \leq c \leq b$ *then* $(c' \cup a) \cap b = (c' \cap b) \cup a$ (*cf.* Beran, 1984; Kalmbach, 1983).

Lemma 7. *If* $\mathcal{L} = (L, \cup, \cap, ', 0, 1)$ *is an orthomodular lattice and* $a, b \in L$ *with* $a \leq b$ *and if one defines*

$$x^* := (x' \cup a) \cap b = (x' \cap b) \cup a$$

for all $x \in [a, b]$ *then* $([a, b], \cup, \cap, ^*, a, b)$ *is an orthomodular lattice, called the* interval-orthomodular lattice $[a, b]$ *induced by* $\mathcal{L}$.

Theorem 5. *If* $\mathcal{L} = (L, \cup, \cap, ', 0, 1)$ *is an orthomodular lattice and* $\mathcal{A} := (L, \cdot, 1)$ *is the corresponding orthomodular implication algebra then the orthomodular lattices* $([p, 1], \vee, \wedge_p, ^p, p, 1)$ *of* $\mathbf{S}(\mathcal{A})$ *are interval-orthomodular lattices induced by* $\mathcal{L}$.

Proof: According to Lemma 3, and Theorem 2, $\cup = \vee$. For all $p \in L$ and all $x \in [p, 1]$ it holds

$$x^p = xp = (x' \cap p') \cup p = x' \cup p = (x' \cup p) \cap 1 = (x' \cap 1) \cup p. \quad \square$$

REFERENCES

Abbott, J. C. (1967). Semi-boolean algebra. *Matematički Vesnik* **4**, 177–198.

Beran, L. (1984). Orthomodular Lattices—Algebraic Approach, D. Reidel, Dordrecht.

Burmeister, P. and Mączyński, M. (1994). Orthomodular (partial) algebras and their representations. *Demonstratio Mathematica* **27**, 701–722.

Diego, A. (1967). Sur les algèbres de Hilbert. *Collection de Logique Mathématique, Sér. A* **21**, 177–198.

Dorninger, D., Länger, H., and Mączyński, M. (2000). Lattice properties of ring-like quantum logics. *International Journal of Theoretical Physics* **39**, 1015–1026.

Kalmbach, G. (1983). Orthomodular Lattices, Academic Press, London.

Pulmannová, S. (1993). Logico-algebraic structures. I: *International Journal of Theoretical Physics* **32**, 1661–1674.